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Comparison of various numerical schemes for simulating fluid flow in variably saturated porous media

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Introduction and Objectives

- Prediction of fluid movement in unsaturated soil using Richards equation
- Nonlinear Richards equation in variably saturated porous media connected with constitutive relations, heterogeneities, irregular geometries, and complex boundary conditions
- Difficulty in solving the Richards equation originated from it's nonlinear characteristic
- To demonstrate a full linearization of the Richards equation with Gardner constitutive relations by using Kirchhoff integral transformation in a transient variably saturated flow
- To show that the integral transformation approach is not only more computationally efficient but also more robustness than other existing numerical methods: h-based model, Celia model (Celia et al., 1990), Kirkland model (Kirkland et al., 1992), and 3DFEMWATER (Yeh et al., 1992)

Mathematical Formulation

Governing Equation

$$\nabla \bullet (K \bullet k_r \nabla (h+z)) + Q = \frac{\partial \theta}{\partial t} \qquad q = -K \bullet k_r \nabla (h+z)$$

Gardner Constitutive Relations

$$\theta = \begin{cases} \theta_r + (\theta_s - \theta_r) e^{\lambda h} & h<0 \\ \theta_s & h \geq 0 \end{cases} \qquad k_r = \begin{cases} e^{\lambda h} & h<0 \\ 1 & h \geq 0 \end{cases}$$

Boundary Condition

$$h = h_0 \quad \Gamma_1 \qquad |q| = |-K \bullet k_r \nabla (h+z)| = q_0 \quad \Gamma_2$$

Kirchhoff Integral Transformation

$$\phi(h) = \int_{-\infty}^{h} k_r(s) ds$$

$$\nabla \bullet (K \bullet \nabla \phi) + \nabla \bullet (k_r K \bullet \nabla z) + Q = \lambda (\theta_s - \theta_r) \frac{\partial \phi}{\partial t} \qquad h<0$$

$$\nabla \bullet (K \bullet \nabla \phi) + \nabla \bullet (K \bullet \nabla z) + Q = 0 \qquad h \geq 0$$

$$\frac{\partial \theta}{\partial h} = \lambda (\theta_s - \theta_r) e^{\lambda h} \qquad \frac{\partial h}{\partial \phi} = \frac{1}{k_r(h)} = \frac{1}{e^{\lambda h}} \qquad \phi(h) = \begin{cases} \frac{1}{\lambda} k_r(h) & h<0 \\ \frac{1}{\lambda} + h & h \geq 0 \end{cases}$$

Linearized Governing Equation and Boundary Conditions

$$\nabla \bullet (K \bullet \nabla \phi) + \nabla \bullet (\lambda \phi K \bullet \nabla z) + Q = \lambda (\theta_s - \theta_r) \frac{\partial \phi}{\partial t} \qquad \phi < \frac{1}{\lambda}$$

$$\nabla \bullet (K \bullet \nabla \phi) + \nabla \bullet (K \bullet \nabla z) + Q = 0 \qquad \phi \geq \frac{1}{\lambda}$$

$$\phi_0 = \phi(h_0) \quad on\ \Gamma_1 \qquad |q| = \begin{cases} |-K \bullet \nabla \phi - \lambda \phi K \bullet \nabla z| = q_0 & \text{for } \phi < \frac{1}{\lambda} \\ |-K \bullet \nabla \phi - K \bullet \nabla z| = q_0 & \text{for } \phi \geq \frac{1}{\lambda} \end{cases} \quad on\ \Gamma_2$$

Numerical Implementation

$$\left\{ \frac{[CK]_{i,j}}{\Delta t} + [DK]_{i,j} + [GK]_{i,j} \right\} \{\phi\}_j^{n+1} = \frac{[CK]_{i,j}}{\Delta t} \{\phi\}_j^n + \{B\}^{n+1}{}_i + \{QG\}_i + \{Q\}_i^{n+1}$$

$$[CK]_{i,j} = \sum_j \int_V N_i \theta_s \lambda (1 - S_{wr}) N_j dV \quad \phi < \frac{1}{\lambda} \qquad [CK]_{i,j} = 0 \quad \phi \geq \frac{1}{\lambda}$$

$$[DK]_{i,j} = \sum_j \int_V \nabla N_i \bullet K \bullet \nabla N_j dV \qquad \{Q\}_i = \int_V N_i Q dV$$

$$[GK]_{i,j} = \sum_j \int_V \lambda K \bullet \nabla N_i \bullet \nabla z N_j dV \quad \phi < \frac{1}{\lambda} \qquad [GK]_{i,j} = 0 \quad \phi \geq \frac{1}{\lambda}$$

$$\{B\}_i = \{B_1\}_i + \{B_2\}_i = \int_\Gamma N_i K \bullet \nabla \phi d\Gamma + \int_\Gamma N_i \lambda \phi K \bullet \nabla z d\Gamma = -\int_\Gamma N_i q d\Gamma \quad \phi < \frac{1}{\lambda}$$

$$\{B\}_i = \{B_1\}_i + \{B_2\}_i = \int_\Gamma N_i K \bullet \nabla \phi d\Gamma + \int_\Gamma N_i K \bullet \nabla z d\Gamma = -\int_\Gamma N_i q d\Gamma \quad \phi \geq \frac{1}{\lambda}$$

$$\{QG\}_i = 0 \quad \phi < \frac{1}{\lambda} \qquad \{QG\}_i = -\int_V \nabla N_i \bullet K \bullet \nabla z dV \quad \phi \geq \frac{1}{\lambda}$$

Numerical Experiments

Example 1

- This example is selected to represent the simulation of a one-dimensional homogeneous problem. The conceptualization of the problem is given in terms of initial and boundary conditions and material properties in Fig. 1. For numerical simulations, the h-based model and Kirchhoff integral transformation method are given. For this problem, water is applied to the top of a vertical soil column at a constant rate of 20 cm/day for about 70 min. The unsaturated characteristic hydraulic properties of the soil in the column are characterized with Gardner relationship.

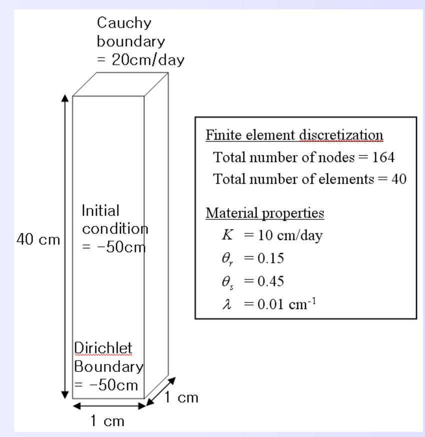

Fig 1. Initial and boundary conditions

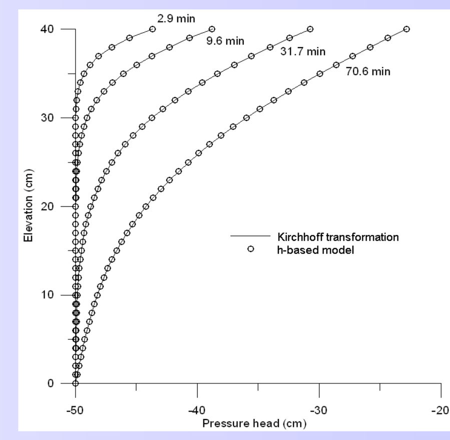

Fig 2. Results of Example 1

Fig 2 illustrates very good agreement between the two methods. CPU with Kirchhoff transformation is faster 1.7 times than with h-based model.

Example 2

For the sixth test, 1-D vertical infiltration in a heterogeneous layered soil column is considered.

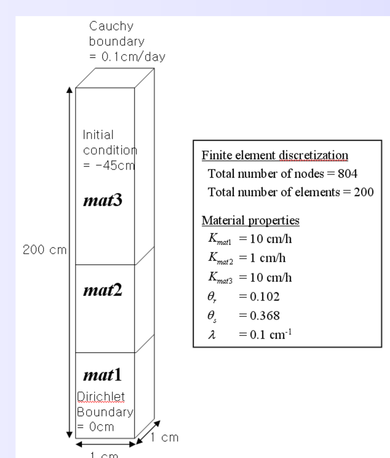

Fig 3. Initial and boundary conditions

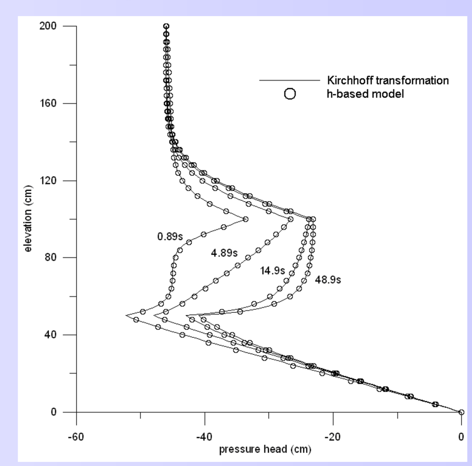

Fig 4. Results of Example 2

Fig 4 illustrates very good agreement between the two methods. The result from Kirchhoff transformation is faster 15.1 times than from h-based model.

Example 3

A base case solution to a Richards equation was calculated for the purpose of comparing and evaluating the six models.

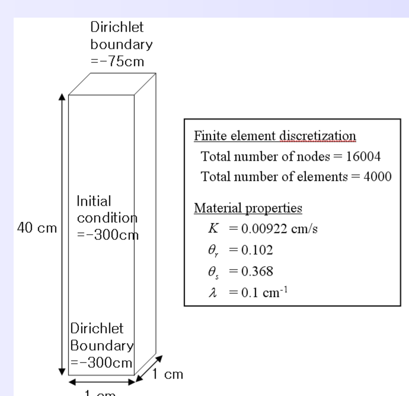

Fig 5. Initial and boundary conditions

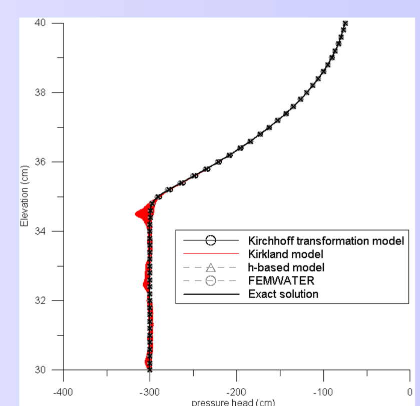

Fig 6. Results of Example 3

Celia model ($\theta - based$ model) and Kirkland model fail to produce converged solution at initial pressure head under -300cm.

Conclusion

- Kirchhoff integral transformation in a transient variably saturated flow makes Richards equation with Gardner constitutive relations full linearized form
- Kirchhoff integral transformation approach is not only more computationally efficient but also more robustness than other existing models

ACKNOWLEDGEMENT

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